AN EFFICIENT HAAR WAVELET TECHNIQUE FOR THE NUMERICAL SOLUTION OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

[1]Sahar Altaf , [2]Sheikh Muhammad ali, [3]Sanjay Kumar
[1]saharaltaf88@gmail.com
[1]College of Humanities and Sciences*PAF-KIET Karachi Institute of Economics and Technology, Karachi
[2]s.m.ali@iobm.edu.pk
[2]College of Computer Science and Information Systems*Institute of Business Management, Karachi
[3]sanjay.kumar@pafkiet.edu.pk
[3]*College of Computer Science* PAF-KIET Karachi Institute of Economics and Technology, Karachi*

***ABSTRACT**. In this paper, an efficient numerical method Haar wavelet has been presented for solving fractional partial differential equations. The fractional derivatives are described in Caputo sense. The approximate solution of some fractional partial differential equations namely Heat equation with lateral heat loss, KDV-type, Klien Gordon and KPP equation with initial –boundary conditions are considered. By using Operational matrix based on Haar wavelet technique, a differential equation is transformed into matrix form of order 2M x 2M which can be solved by using MATLAB and then compared with the exact solution. The suggested technique is simple and effective for solving fractional partial differential equations numerically.*

Key words: Fractional Partial differential equation, Caputo derivative, Haar Wavelet

1 INTRODUCTION

Partial differential equations of arbitrary order (FPDEs) are standard form of classical order partial differential equations, extensively applied in the modeling of fluid flow problems, viscoelasticity, natural science, material science, engineering and many other fields. Fractional derivatives are an exceptional tool for the explanation of remembrance and inherited properties of a range of resources and processes [1-3]. Presently no technique exists by which an exact solution for fractional partial differential equation can be obtained. On this basis numerical solutions are derived for the solution of fractional partial differential equations. Numerous existing numerical techniques for fractional order differential equations are finite difference scheme [20], Adomian decomposition technique [21], Homotopy perturbation method [22], Differential Transform Method [23], Simplest equation method [24], Reduced Differential Transform Method [25] and Homotopy Analysis Method [26].

Wavelets techniques are also used to find solutions of PDEs numerically. They are used to detect signals and processing of an image. In the start of the early 1990s, wavelet techniques have given much attention to solve PDEs [13]. During the preceding two decades this problem has attracted great concentration and frequent papers about this topic are available, for instances see [4-11]. Chen and Hsiao [27] were first to put Haar function into focus to solve a differential equation. They determined the operational matrix of integrals and applied Haar wavelet into dynamic models such as lumped and distributed-parameter models.

Few researchers obtained solutions of fractional PDEs by Haar wavelet like Burger-Fisher and generalized Fisher equation [28], fractional Fokker Plank equation [29], fractional Benny equation [30]. The aim of the present work is to using Haar wavelets with operational matrix of fractional integration for numerical solution of the above mentioned FPDEs. The method is simple and accurate for small number of collocation points. The outline of this manuscript is as follows. In section 2, we describe some basic definition and properties of fractional calculus. In section 3, we describe Haar wavelets and its approximation. In Section 4, the proposed method and its implementation to solve the aforesaid problems is given. In section 5, the numerical examples are presented. Finally, a conclusion is drawn in section 6.

2 FRACTIONAL CALCULUS

Numerous different definitions on fractional derivative are available. Some of these are Riemann-Liouville, Grunwald-Letnikow, Caputo etc. The most frequently in use are the Riemann-Liouville and Caputo derivative.

Definition 2.1 [19] Fractional Riemann-Liouville integral of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau) d\tau \qquad (2.1)$$

Some of its properties are as follows. For $f \in C_\mu$, $\mu \geq -1$, α , $\beta \geq 0$, and $\gamma > 1$:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
3. $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

Definition 2.2 Fractional derivative of function f(x) in Caputo sense is defined as [19]

$$D_*^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_o^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \qquad (2.2)$$

For $m-1 < \alpha < m$, $m \in N$, $x > 0$, $f \in C_{-1}^m$

Some Properties

1. [14] If $\alpha \geq 0$ and $f(t) = (t-a)^{\beta}$, $m = \lceil \alpha \rceil$, then

$$ {}^{c}D_{a}^{\alpha} f(t) = \begin{cases} 0 & if\ \beta \in \{0,1,2,\dots,m-1\}, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, & if\ \beta \in N,\ and\ \beta \geq m\ or \\ & \beta \notin N,\ and\ \beta > m-1 \end{cases} \tag{2.3} $$

2. [14] Let $\alpha > 0$, $m = \lceil \alpha \rceil$ *and* $f \in AC^{m}[a,b]$, then

$$ I_{a}^{\alpha}\left[{}^{c}D_{a}^{\alpha} f(t)\right] = f(t) - \sum_{j=0}^{m-1} \frac{D^{j} f(a)}{j!}(t-a)^{j} \tag{2.4} $$

3 HAAR WAVELET

For $t \in [0, 1]$, Haar wavelet functions are defined by [12]

$$ h_0(t) = 1, \tag{3.1} $$

$$ h_i(t) = \begin{cases} 1, & \dfrac{k}{m} \leq t < \dfrac{k+0.5}{m} \\ -1 & \dfrac{k+0.5}{m} \leq t < \dfrac{k+1}{m} \\ 0 & otherwise, \end{cases} \tag{3.2} $$

Here $i = 0, 1, 2, \dots\dots m-1$, $m = 2^{j}$, $j \geq 0$, $0 \leq k \leq 2^{j}-1$, *j and k* corresponds to integer decomposition of the index i, $i = 2^{j} + k - 1$, $j \geq 0$. Maximum of i is $M = 2m = 2^{J+1}$

Function approximation

A function $u(t)$ can be extended into Haar wavelet by [12]

$$ u(t) = \sum_{i=0}^{\infty} d_i h_i(t), \quad where\ \ d_i = \int_0^1 u(t) h_i(t)\,dt \tag{3.3} $$

Approximating $u(t)$ as piecewise constant from beginning to end in subintervals, Eq. (4.3) will be concluded at fixed terms

$$ u(t) \approx \sum_{i=0}^{m-1} d_i h_i(t) = d^{T} H_m(t), \tag{3.4} $$

where

$d = [d_0, d_1, \dots\dots, d_{m-1}]^{T}$, $H_m(t) = [h_0(t), h_1(t), \dots. h_{m-1}(t)]^{T}$,

m is a power of 2.

The matrix from of Eq. (3.4) is

$$ u = d^{T} H, \tag{3.5} $$

Where the row vector u is the discrete form the function $u(t)$. H is Haar wavelet matrix of order $m = 2^{j}$, j= 0, 1, 2, ….. .J, i.e.

$$ H = \begin{bmatrix} h_0(t_0) & h_0(t_1) & \cdots & h_0(t_{m-1}) \\ h_1(t_0) & h_1(t_1) & \cdots & h_1(t_{m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ h_{m-1}(t_0) & h_{m-1}(t_1) & \cdots & h_{m-1}(t_{m-1}) \end{bmatrix} \tag{3.6} $$

For arbitrary function u(x,t) $\in L^{2}$ ([0,1) x [0, 1)), can be expanded into Haar series by [12]

$$ u(x,t) \approx \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} u_{ij} h_i(x) h_j(t), \tag{3.7} $$

Where $u_{ij} = \int_0^1 h_i(x) h_j(x)\,dx.$

Eq. (3.7) will be written as

$$ u(x,t) \approx H_m^{T}(x)\, U\, H_m(t) \tag{3.8} $$

In this research, we will apply wavelet collocation method to resolve the coefficients u_{ij}. These collocation points are shown in the following.

$$ x_i = t_i = (2i-1)/2m,\ i = 1, 2, \dots\dots\dots, m \tag{3.9} $$

Discreting Eq.(3.8) by the step Eq. (3.9), we obtain the matrix form of Eq. (3.8)

$$ D = H^{T} U H, \tag{3.10} $$

Where $U = [u_{ij}]_{mxm}$, $D = [u(x_i, t_j)]_{mxm}$

We defined the m-square Haar matrix $\overline{\Phi}_{mxm}$ as [12]:

$$ \overline{\Phi}_{mxm} \overset{\Delta}{=} \left[H_m\left(\frac{1}{2m}\right) H_m\left(\frac{3}{2m}\right) \dots\dots H_m\left(\frac{2m-1}{2m}\right) \right] \tag{3.11} $$

For example, when m = 8, the Haar matrix is expressed as

$$ \overline{\Phi}_{8x8} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{3.12} $$

From the definition of Haar wavelet functions, we may know that H is an orthogonal matrix [12].

Operational matrix of fractional order integration

Here in this part, we may simply begin with the operational matrix of fractional integration of Haar wavelet [15]

First defininig a set of m-term Block Pulse Functions as follows:

$$ \overline{\overline{b}}_i(t) = \begin{cases} 1, & i/m \leq t < (i+1)/m, \\ 0, & otherwise \end{cases} \tag{3.13} $$

where i = 0, 1, 2 ,……. $(m-1)$,

The functions $\overline{b}_i(t)$ are orthogonal. That is,

$$ \overline{\overline{b}}_i(t)\, \overline{\overline{b}}_l(t) = \begin{cases} 0, & i \neq l, \\ \overline{\overline{b}}_i(t), & i = l, \end{cases} \tag{3.14} $$

$$ \int_0^1 \overline{\overline{b}}_i(\tau)\, \overline{\overline{b}}_l(\tau)\, d\tau = \begin{cases} 0, & i \neq l, \\ 1/m, & i = l \end{cases} \tag{3.15} $$

Since Haar wavelet functions are piecewise constant, extending it into an m-term block pulse functions as

$$ H_m(t) = \overline{\Phi}_{mxm} \overline{\overline{B}}_m(t) \tag{3.16} $$

Where $\overline{\overline{B}}_m(t) \overset{\Delta}{=} \left[\overline{\overline{b}}_0(t)\ \overline{\overline{b}}_1(t) \dots\dots \overline{\overline{b}}_i(t)\ \dots\dots\ \overline{\overline{b}}_{m-1}(t) \right]^{T}$

The Block Pulse operational matrix of the fractional order, integration $\overline{F}^{\alpha}$ proposed by Kilicman [10] is as follows:

$$ \left(I^{\alpha} \overline{\overline{B}}_m\right)(t) \approx \overline{F}^{\alpha} \overline{\overline{B}}_m(t) \tag{3.17} $$

where

$$\overline{F}^{\alpha} = \frac{1}{m^{\alpha}} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3.18)$$

with $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$

Now, deriving the Haar wavelet operational matrix of the fractional order

Taking,

$$\left(I^{\alpha} H_m\right)(t) \approx P^{\alpha}_{mxm} H_m(t) \qquad (3.19)$$

where the square matrix of order m P^{α}_{mxm} is called the Haar wavelet operational matrix of the fractional order integration. Using Eqs. (3.16) and (3.17), we have.

$$\left(I^{\alpha} H_m\right)(t) \approx \left(I^{\alpha} \overline{\Phi}_{mxm} \overline{\overline{B}}_m\right)(t) = \overline{\Phi}_{mxm} \left(I^{\alpha} \overline{\overline{B}}_m\right)(t)$$
$$\approx \overline{\Phi}_{mxm} \overline{F}^{\alpha} \overline{\overline{B}}_m(t) \qquad (3.20)$$

From Eqs. (3.19) and (3.20) we get

$$P^{\alpha}_{mxm} H_m(t) = P^{\alpha}_{mxm} \overline{\Phi}_{mxm} \overline{\overline{B}}_m(t) = \overline{\Phi}_{mxm} \overline{F}^{\alpha} \overline{\overline{B}}_m(t) \qquad (3.21)$$

Then, the Haar wavelet operational matrix of the fractional order integration P^{α}_{mxm} is given by [15]

$$P^{\alpha}_{mxm} = \overline{\Phi}_{mxm} \overline{F}^{\alpha} \overline{\Phi}^{-1}_{mxm} \qquad (3.22)$$

Let, $\alpha = 0.5$, m=8, the operational matrix P^{α}_{mxm} is computed below:

$$P^{0.75}_{8x8} = \begin{bmatrix} 0.7523 & -0.2203 & -0.1558 & -0.0820 & -0.1102 & -0.0580 & -0.0447 & -0.0377 \\ 0.2203 & 0.3116 & -0.1558 & 0.2296 & -0.1102 & -0.0580 & 0.1756 & 0.0782 \\ 0.0410 & 0.1148 & 0.2203 & -0.0350 & -0.1102 & 0.1623 & -0.0389 & -0.0063 \\ 0.0779 & -0.0779 & 0 & 0.2203 & 0 & 0 & -0.1102 & 0.1623 \\ 0.0094 & 0.0196 & 0.0812 & -0.0032 & 0.1558 & -0.0247 & -0.0026 & -0.0009 \\ 0.0112 & 0.0439 & -0.0551 & -0.0194 & 0 & 0.1558 & -0.0247 & -0.0026 \\ 0.0145 & -0.0145 & 0 & 0.0812 & 0 & 0 & 0.1558 & -0.0247 \\ 0.0275 & -0.0275 & 0 & -0.0551 & 0 & 0 & 0 & 0.1558 \end{bmatrix}$$

Similarly, we can find P^{α} for different values of α. This operational matrix of fractional order integration is used in the proposed method by which numerical solution of partial differential equations has been obtained for different values of α as discussed in section 5.

4 IMPLEMENTATION OF THE METHOD

Consider a generalized fractional PDE

$$\frac{\partial^{\alpha} w(x,t)}{\partial t^{\alpha}} + \lambda \frac{\partial^{\beta} w(x,t)}{\partial t^{\beta}} + \mu\, w(x,t) = \eta \frac{\partial^{\gamma} w(x,t)}{\partial x^{\gamma}} + f(x,t) \quad (5.1)$$

Approximating $\frac{\partial^{\alpha} w(x,t)}{\partial t^{\alpha}}$ by 2D Haar wavelet series as

$$\frac{\partial^{\alpha} w(x,t)}{\partial t^{\alpha}} = \overline{\Phi}^{T}_{m}(x) D \overline{\Phi}_m(t) \qquad (5.2)$$

Applying on both sides of equation (5.2) via fractional integral I^{α}_{t} in variable t, and using p2 [14], we have

$$w(x,t) = \overline{\Phi}^{T}_{m}(x) D P^{\alpha}_{m\times m} \overline{\Phi}_m(t) + a(x)t + b(x) \qquad (5.3)$$

By using the initial conditions $w(x,0) = \phi(x)$, $\left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = \Omega(x)$, Equation (5.3), includes $b(x) = \phi(x)$ and $a(x) = \Omega(x)$. Thus, the equation (5.3) will be

$$w(x,t) = \overline{\Phi}^{T}_{m}(x) D P^{\alpha}_{m\times m} \overline{\Phi}_m(t) + \Omega(x)t + \phi(x) \qquad (5.4)$$

Substituting (5.3), (5.4) in (5.1), we get

$$\eta \frac{\partial^{\gamma} w(x,t)}{\partial t^{\gamma}} = \overline{\Phi}^{T}_{m}(x) D \overline{\Phi}_m(t) + \lambda \overline{\Phi}^{T}_{m}(x) D P^{\alpha-\beta}_{m\times m} \overline{\Phi}_m(t) +$$
$$\mu \overline{\Phi}^{T}_{m}(x) D P^{\alpha}_{m\times m} \overline{\Phi}_m(t) + g(x,t)$$
$$= \overline{\Phi}^{T}_{m}(x) \left\{ D\left(I + \lambda P^{\alpha-\beta}_{m\times m} + \mu P^{\alpha}_{m\times m}\right) + J_{m\times m} \right\} \overline{\Phi}_m(t) \qquad (5.5)$$

where

$$g(x,t) = \Omega(x)\left(\frac{\lambda t^{1-\beta}}{\Gamma(2-\beta)} + \mu t\right) + \mu\phi(x) - f(x,t)$$
$$= \overline{\Phi}^{T}_{m}(x) J_{m\times m} \overline{\Phi}_m$$

Applying I^{γ}_{x} on both sides of equation (5.5), we have

$$\eta w(x,t) = I^{\gamma}_{x} \overline{\Phi}^{T}_{m}(x) \left\{ D\left(I + \lambda P^{\alpha-\beta}_{m\times m} + \mu P^{\alpha}_{m\times m}\right) + J_{m\times m} \right\} \overline{\Phi}_m(t) + x\varphi_1(t) + \varphi_2(t) \qquad (5.6)$$

Incorporating the boundary conditions $w(0,t) = \xi(t)$, we have $\varphi_2(t) = \xi(t)$, and $u(1,t) = \delta(t)$ implies

$$\varphi_1(t) = -I^{\gamma}_{x} \overline{\Phi}^{T}_{m}(1) \left\{ D\left(I + \lambda P^{\alpha-\beta}_{m\times m} + \mu P^{\alpha}_{m\times m}\right) + J_{m\times m} \right\}$$
$$\overline{\Phi}_m(t) + \delta(t) - \xi(t)$$

Substituting (5.7) in (5.6) we have

$$\eta w(x,t) = \overline{\Phi}^{T}_{m}(x) \left(\left(P^{\gamma}_{m\times m}\right)^{T} - \left(Q^{\gamma}_{m\times m}\right)^{T} \right) \left\{ \begin{matrix} D\left(\begin{matrix} I + \lambda P^{\alpha-\beta}_{m\times m} + \\ \mu P^{\alpha}_{m\times m} \end{matrix}\right) \\ + J_{m\times m} \end{matrix} \right\}$$
$$\overline{\Phi}_m(t) + x\left(\delta(t) - \xi(t)\right) + \xi(t)$$

where

$$I^{\gamma}_{x} \overline{\Phi}_m(x) = P^{\gamma}_{m\times m} \overline{\Phi}_m(x) = \overline{\Phi}^{T}_{m}(x) \left(P^{\gamma}_{m\times m}\right)^{T} \text{ and}$$
$$x I^{\gamma}_{x} \overline{\Phi}^{T}_{m}(1) = Q^{\gamma}_{m\times m} \overline{\Phi}_m(x)$$

$$w(x,t) = \frac{1}{\eta}[\overline{\Phi}^{T}_{m}(x) \left(\left(P^{\gamma}_{m\times m}\right)^{T} - \left(Q^{\gamma}_{m\times m}\right)^{T} \right) \left\{ \begin{matrix} D\left(\begin{matrix} I + \lambda P^{\alpha-\beta}_{m\times m} \\ + \mu P^{\alpha}_{m\times m} \end{matrix}\right) \\ + J_{m\times m} \end{matrix} \right\}$$
$$\overline{\Phi}_m(t) + x\left(\delta(t) - \xi(t)\right) + \xi(t)]$$

Equating (5.7) and (5.13)

$$\frac{1}{\eta}[\overline{\Phi}^{T}_{m}(x) \left(\left(P^{\gamma}_{m\times m}\right)^{T} - \left(Q^{\gamma}_{m\times m}\right)^{T} \right) \left\{ D\left(I + \lambda P^{\alpha-\beta}_{m\times m} + \mu P^{\alpha}_{m\times m}\right) + J_{m\times m} \right\}$$
$$\overline{\Phi}_m(t) + x\left(\delta(t) - \xi(t)\right) + \xi(t)] - \overline{\Phi}^{T}_{m}(x) D P^{\alpha}_{m\times m} \overline{\Phi}_m(t)$$
$$-\Omega(x)t - \phi(x) = 0$$

With the help of MATLAB command f solve, we can solve the unknown matrix ‘x’ which is known as the coefficient matrix. This coefficient matrix can be used to find the approximate solution of the above FPDEs given in section 1.

5 NUMERICAL APPLICATIONS

Example 5.1 Consider the heat equation

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \bar{k}\frac{\partial^2 w(x,t)}{\partial x^2} - cw(x,t) + h(x,t), \quad 0 < x < 1, \ t \geq 0,$$

with $\bar{k} = 1,\ c = 0 \text{ and } h(x,t) = 0$ [16].

The initial and boundary conditions are given $w(x,0) = \sin x, w(0,t) = 0 \quad and \quad w(1,t) = \sin(1)e^{-t}$ where $w(x,t)$ measures the temperature of the rod at time t, $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes Caputo fractional derivative of order α , $\bar{k}$ denotes thermal diffusivity that measures the ability of rod to conduct heat, c is a positive constant and $h(x,t)$ is called the heat source. For α =1, the equation reduces to classical heat equation.

The exact solution of the above problem for α = 1, is $w(x,t) = \sin(x)e^{-t}$ [16]. The approximate solutions for cases α = 0.25, 0.5, 0.75, 1.0 are obtained by the aforesaid method. Figure 5.1 shows the numerical and exact solutions for α = 1. The numerical results are also given in Table 5.1. It can be clearly seen that the solution obtained from fractional cases are approaching to classical heat equation.

Table 5.1: Numerical solution of Heat equation for different values of α

x_i/t_i	u_{Haar}				u_{Exact}	Absolute Error
	α= 0.25	α = 0.5	α= 0.75	α = 1		
1/16	0.0517	0.0534	0.0558	0.0585	0.0587	$2.0\text{x}10^{-4}$
3/16	0.1389	0.1415	0.1446	0.1512	0.1545	$3.3\text{x}10^{-3}$
5/16,5/16	0.2074	0.2110	0.2146	0.2173	0.2249	$7.6\text{x}10^{-3}$
7/16,7/16	0.2584	0.2616	0.2640	0.2651	0.2735	$8.4\text{x}10^{-3}$
9/16,9/16	0.2929	0.2951	0.2959	0.2940	0.3039	$9.9\text{x}10^{-3}$
11/16,11/16	0.3118	0.3126	0.3119	0.3090	0.3191	$1.0\text{x}10^{-2}$
13/16,13/16	0.3153	0.3153	0.3143	0.3126	0.3222	$9.6\text{x}10^{-3}$
15/16,15/16	0.3039	0.3041	0.3047	0.3058	0.3157	$9.9\text{x}10^{-3}$

Example 5.2 Consider the fractional Klien-Gordon equation

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} - \frac{\partial^2 w(x,t)}{\partial x^2} + w(x,t) = h(x,t), \quad 1 < \alpha \leq 2$$

subject to

$$w(x,0) = 0 \quad \frac{\partial w(x,0)}{\partial t} = 0, \quad and \quad w(0,t) = 0 \quad w(1,t) = t^3$$

$$and \quad h(x,t) = 6x^3 t + x^3 t^3 - 6xt^3$$

The exact solution for α = 2 of eq. (5.2) is

$w(x,t) = x^3t^3$[17].

Plots of numerical solution for different values of α are shown in Figure 5.2.

Example5.3 Consider the non-homogeneous fractional third order dispersive partial differential equation [18]

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} + \frac{\partial^3 w(x,t)}{\partial x^3} = f(x,t), \quad 0 < x < 1, t > 0 \qquad (5.18)$$

where $f(x,t) = -\sin \pi x \sin t - \pi^3 \cos \pi x \cos t$

subject to initial condition and time- dependent boundary conditions

$$w(x,0) = \sin \pi x$$

$$w(0,t) = 0, \ w_x(0,t) = \pi \cos t, \ w_{xx}(0,t) = 0$$

The exact solution is $w(x,t) = \sin \pi x \cos t$ [18]. The numerical results are also given in Table 5.2 for different values of α. It can be clearly seen that the solution obtained from fractional cases is approaching to the classical order equation. Figure 5.3 shows a numerical and exact solution for α = 1.

Table 5.3 Numerical and Exact solutions for different values of α

x_i/t_i	u_{Haar}				u_{Exact}	Absolute Error
	α= 0.5	α=0.75	α=0.95	α = 1		
1/16,1/16	0.1947	0.1947	0.1947	0.1947	0.1947	0.0000
3/16,3/16	0.5458	0.5459	0.5459	0.5459	0.5458	$1.0\text{x}10^{-4}$
5/16,5/16	0.7911	0.7912	0.7913	0.7913	0.7912	$1.0\text{x}10^{-4}$
7/16,7/16	0.8877	0.8883	0.8889	0.8891	0.8884	$7.0\text{x}10^{-4}$
9/16,9/16	0.8275	0.8290	0.8301	0.8303	0.8297	$6.0\text{x}10^{-4}$
11/16,11/16	0.6378	0.6409	0.6438	0.6446	0.6426	$2.0\text{x}10^{-3}$
13/16,13/16	0.3736	0.3790	0.3830	0.3838	0.3821	$1.8\text{x}10^{-3}$
15/16,15/16	0.1031	0.1113	0.1170	0.1179	0.1155	$2.4\text{x}10^{-3}$

Example 5.4 Consider the fractional KPP equation of the form [19]

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^\beta w(x,t)}{\partial x^\beta} - w(x,t) \quad 0 < \alpha \leq 1, 1 < \beta \leq 2 \quad \text{subject to}$$

some initial and boundary condition

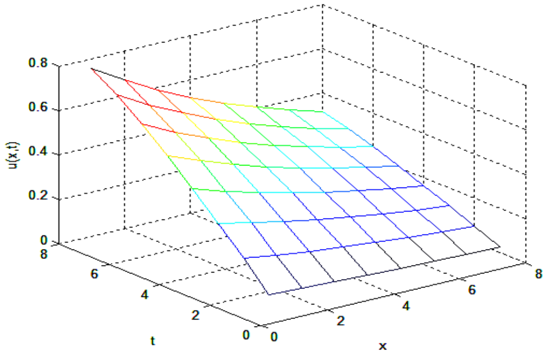
$$w(x,0) = x + e^{-x}$$

$$w(0,t) = 1, \qquad \frac{\partial w(0,t)}{\partial t} = e^{-t} - 1$$

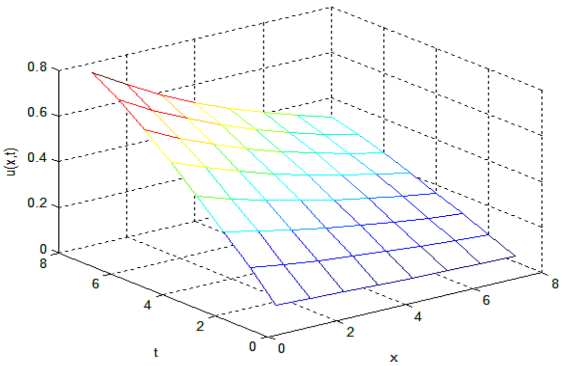
For α = 1, the exact solution of the problem is $w(x,t) = xe^{-t} + e^{-x}$. Plots of numerical and exact solution for fractional values of α are shown in Figure 5.4

CONCLUSION

This manuscript presented numerical results of well known Heat equation with lateral heat loss, KDV-type and Klien Gordon and KPP equation by using Haar wavelets together with their operational matrix of fractional order integration. The results show the accuracy of the proposed method. Haar technique yields worthy results for small values of m (i.e m=8). For larger values of m (i.e m=16, m=32, m=64, m=128), the results are closer to real values. The method is very suitable for solving boundary value problems, since the boundary condition i taken report automatically. Also the

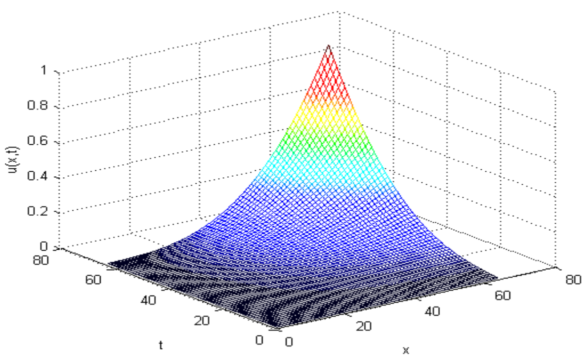


a) Numerical solution of Heat equation for $\alpha = 1$

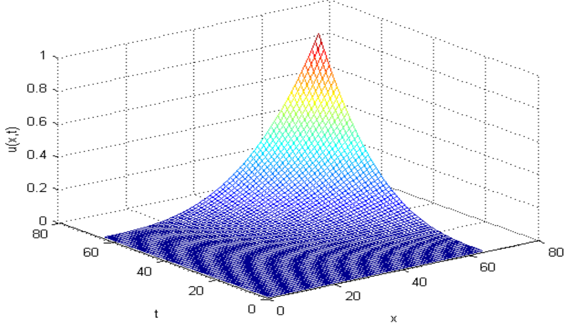


b) Exact solution of Heat equation for α = 1

Figure 5.1: Solutions for Heat Equation for α = 1

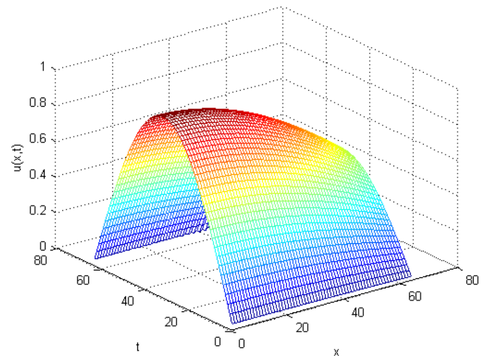


a) Exact solution of Klien Gordon equation for $\alpha = 2$

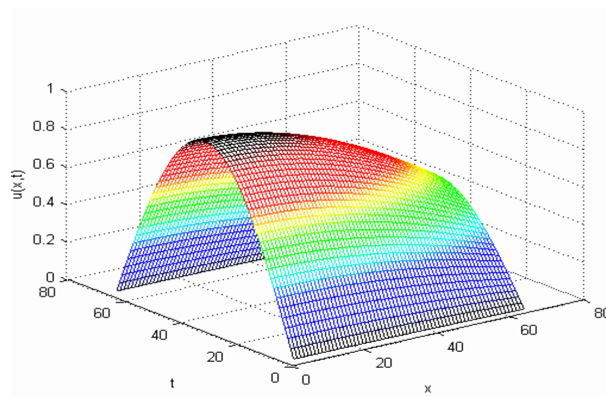


b) Numerical solution of Klien-Gordon equation for $\alpha = 1.75$

Figure 5.2 Solution of Klien-Gordon Equation for $\alpha = 2$ and α =1.75



a) Numerical solution of KDV equation for $\alpha = 1$



b) Exact solution of KDVequation for $\alpha = 1$

Figure 5.3 Numerical and Exact solution of KDV-type Equation for $\alpha = 1$

anticipated method is easy in execution, simple and having low computation costs. It can be used for other kinds of fractional partial differential equations.

ACKNOWLEDGEMENT

The authors are very thankful to PAF-Karachi Institute of Economics and Technology for the financial assistance.

REFERENCES

[1] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[2] K. B. Oldham and J. Spanier, The Fractional Calculus. Academic Press, New York, 1974.

[3] I. Podlubny, Fractional Differential Equations. Academic Press, San Diego, 1999.

[4] C. Cattani, Haar wavelets based technique in evolution problems, Chaos,Proc. Estonian Acad. Sci. Phys. Math, **1**, 45–63(2004).

[5] G. Hariharan, K. Kannan, and K.R. Sharma, Haar wavelet method for solving fishers equation, Appl. Math. Comput, **211**(2), 284–292(2009).

[6] Z. Chun and Z. Zheng, Three-dimensional analysis of functionally graded plate based on the haar wavelet method, Acta. Mech. Solida. Sin.,**20**(2), 95–102(2007)

[7] J. Majak, M. Pohlak, M. Eerme, and T. Lepikult,Weak formulation based haar wavelet method for solving differential equations, Appl. Math.
Comput., **211**, 488–494(2009).

[8] M. H. Heydari, M. R. Hooshmandasl, M. F. Maalek Ghaini and F. Fereidouni, Two-dimensional Legendre wavelets for solving fractional Poisson equation with Dirichlet boundary conditions Engineering Anal. Boun. Elem., **37**,1331-1338(2013).

[9] M. U. Rehman and R. A. Khan, The legendre wavelet method for solving fractional differential equations, Commun. Nonlinear Sci. Numer. Simul., **227**(2), 234–244(2009).

[10] A. Kilicman and Z.A. Al Zhour, Kronecker operational matrices for fractional calculus and some applications, Appl. Math. Comput., **187**(1), 250–65(2007).

[11] M. H. Heydari, M. R. Hooshmandasl, F. M. Maalek Ghaini and F. Mohammadi, Wavelet Collocation Method for Solving Multiorder Fractional Differential Equations, J. Appl. Math., vol. 2012, 19, 2012.

[12] Wang, L., Ma, Y., & Meng, Z. Haar wavelet method for solving fractional partial differential equations numerically. Applied Mathematics and Computation, **227**, 66-76(2014)

[13] U. Lepik, Solving pdes with the aid of two-dimensional Haar wavelets, Comput. Math. Appl., **61**, 1873–1879(2011).

[14] Rehman, M. & Khan, R. Numerical solutions to initial and boundary value problems for linear fractional partial differential equations. Applied Mathematical Modelling, **37**(7), 5233-5244(2013).

[15] Li, Y. & Zhao, W. Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations. Applied Mathematics and Computation, **216**(8), 2276-2285(2010).

[16] Heydari, M. H., Maalek Ghaini, F. M., & Hooshmandasl, M. R. Legendre wavelets method for numerical solution of time-fractional heat equation. Wavelet and Linear Algebra, **1**(1), 19-31(2014).

[17] Odibat, Z. & Momani, S. The Variational Iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics. Computers & Mathematics with Applications, **58**(11-12), 2199-2208(2009).

[18] Ravi Kanth, A. & Aruna, K. Solution of fractional third-order dispersive partial differential equations. Egyptian Journal of Basic and Applied Sciences, **2**(3), 190-199(2015).

[19] Brikaa, M. An analytic algorithm for the space-time fractional reaction-diffusion equation. Journal of Interpolation and Approximation in Scientific Computing, (2), 112-127(2015).

[20] Meerschaert, M. & Tadjeran, C. Finite difference approximations for fractional Advection–Dispersion flow equations. Journal of Computational and Applied Mathematics, **172**(1), 65-77(2004).

[21]. Jafari, H. & Daftardar-Gejji, V. Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition. Applied Mathematics and Computation, **180**(2), 488-497(2006).

[22]. Taghizadeh, N., Mirzazadeh, M., Rahimian, M., & Akbari, M. Application of the simplest equation method to some time-fractional partial differential equations. Ain Shams Engineering Journal, **4**(4), 897-902(2013).

[23] Tadjeran, C., Meerschaert, M., & Scheffler, H. A second-order accurate numerical approximation for the fractional diffusion equation. Journal of Computational Physics, **213**(1), 205-213(2006).

[24]. Srivastava, V., Awasthi, M., & Tamsir, M. RDTM solution of Caputo time fractional-order hyperbolic telegraph equation. AIP Advances, **3**(3), 032142(2013).

[25]. Odibat, Z. & Momani, S. A generalized differential transform method for linear partial differential equations of fractional order. Applied Mathematics Letters, **21**(2), 194-199(2008).

[26]. Dehghan, M., Manafian, J., & Saadatmandi, A. The solution of the linear fractional partial differential equations using the Homotopy Analysis method. Zeitschrift für Naturforschung-A, **65**(11), 935(2010).

[27]. Chen, C. & Hsiao, C. Haar Wavelet method for solving lumped and distributed-parameter systems. IEEE Proceedings - Control Theory and Applications, **144**(1), 87-94(1997).

[28]. Gupta, A. & Ray, S. On the solutions of Fractional Burgers-Fisher and Generalized Fisher's Equations using two reliable methods. International Journal of Mathematics and Mathematical Sciences, 1-16 (2014)

[29]. Saha Ray, S. & Gupta, A. A two-dimensional Haar Wavelet approach for the numerical simulations of time and space fractional Fokker–Planck equations in modelling of anomalous diffusion systems. Journal of Mathematical Chemistry, **52**(8), 2277-2293(2014).

[30]. Akinlar, M., Secer, A., & Bayram, M. Numerical Solution of Fractional Benney Equation. Appl. Math. Inf. Sci., **8**(4), 1633-1637(2014).